March 25: Nagata Rings, part 3

Nagata implies analytically unramified

The next important theorem shows that a Nagata local domain is analytically unramified.

Theorem H. Assume that $(R, \mathfrak{m}, k)$ is a local Nagata ring. Then R is analytically unramified.

Proof. We proceed with the following steps.

Step 1. Reduction to the case that R is integrally closed.

Proof. We use the fact that if A is a semi-local ring with maximal ideals $P_1, \dots, P_c$ and $J = P_1 \cap \cdots \cap P_c$, then the J-adic completion of A is the direct sum of the P_i-adic completions of A.

To see this, note that for each n,

$$A/J^n \cong A/P_1^n \oplus \cdots \oplus A/P_c^n.$$

Now use the fact that inverse limits commute with direct sums to conclude $\widehat{A}^J = \widehat{A}^{P_1} \oplus \cdots \oplus \widehat{A}^{P_c}$. Note that in this case, $\widehat{A}^J$ is reduced if and only if each $\widehat{A}^{P_i}$ is reduced.

Nagata implies analytically unramified, continued

We apply the foregoing to R'. Since R' is finite over R, R has finitely many maximal ideals and $\widehat{R'} = R' \otimes \widehat{R}$, hence the inclusion $R \otimes \widehat{R} \to R' \otimes \widehat{R}$ shows that $\widehat{R}$ is contained in the completion of R' with respect to $\mathfrak{m}R'$.

On the other hand, $\sqrt{\mathfrak{m}R'} =: J$ is the Jacobson radical of R'. Thus, the completions of R' with respect to $\mathfrak{m}R'$ and J are the same. The latter is the direct sum of the completions of R' with respect to $P_1, \ldots, P_c$, where the P_i are the maximal ideals of R'.

If each $\widehat{R'}^{P_i}$ is reduced, then $\widehat{R}$ is reduced, which is what we want.

Since each R'_{P_i} is also a Nagata ring and $\widehat{R'}^{P_i} = \widehat{R'_{P_i}}^{P_i}$, it suffices to show each R'_{P_i} is analytically unramified. Thus, we may now assume that R is integrally closed.

Nagata implies analytically unramified, continued

Step 2. If $0 \neq x \in R$ and $P \in \mathrm{Ass}(\widehat{R}/x\widehat{R})$ satisfies $R/(P \cap R)$ is analytically unramified, then $(\widehat{R})_P$ is a DVR.

Proof. Set $P_0 := P \cap R$ and localize R at P_0. We note that $P_0 \in \mathrm{Ass}(R/xR)$. If not P_0, contains a non-zero divisor on R/xR, which remains a non-zero divisor on the faithfully flat extension $(\hat{R}/x\hat{R})_{P_0}$, contrary to our assumption on P.

Thus, R_{P_0} is a DVR (by Proposition A) and $P_0 = yR$, for some $y \in R$. Since $\widehat{R}$ is (still) flat over R, y is a non-zerodivisor in $\widehat{R}$.

It follows that $P \in \mathrm{Ass}(\widehat{R}/y\widehat{R}) = \mathrm{Ass}(\widehat{R}/P_0\widehat{R})$.

On the other hand, $\widehat{R}/P_0\widehat{R}$ is reduced (by assumption). Thus

$$(\widehat{R}/P)_P = (\widehat{R}/P_0\widehat{R})_P = (\widehat{R}/y\widehat{R})_P.$$

Therefore, P_P is principal, so $\widehat{R}_P$ is a DVR.

Nagata implies analytically unramified, continued

Step 3. We prove the theorem by induction on the dimension of R.

Proof. Suppose R has dimension one and take $0 \neq x \in R$. Since R' is finite over R, there exists $k \geq 1$ such that $x^nR' \cap R \subseteq x^{n-k}R$, for all $n \geq k$. Thus $\overline{x^nR} \subseteq x^{n-k}R$, for all $n \geq k$. Since xR is $\mathfrak{m}$-primary, R is analytically unramified by Rees's theorem.

If R has dimension greater than one, then by induction, R/Q is analytically unramified for all non-zero prime ideals $Q \subseteq R$. Fix $0 \neq x \in R$, and take $P \in \mathrm{Ass}(\widehat{R}/x^n\widehat{R})$. By Steps 1 and 2, $\widehat{R}_P$ is a DVR. By Proposition A, $x^n\widehat{R}$ is integrally closed, for all $n \geq 1$.

Since the nilradical of $\widehat{R}$ is contained in the integral closure of every ideal, the nilradical of $\widehat{R}$ is contained in $x^n\widehat{R}$ for all n.

Thus the nilradical of $\widehat{R}$ is zero, which completes the proof. □

R satisfies N_2 implies R[x] satisfies N_2

We need one more component, of independent interest, before we can prove the main result of this section. For this result, we will use the following fact about polynomial rings, whose proof we leave as an exercise. Let $A \subseteq B$ be commutative rings and $f(x) \in B[x]$.

Then $f(x)$ is integral over $A[x]$ if and only if each coefficient of $f(x)$ is integral over A. It follows that if A is an integrally closed integral domain, the $A[x]$ is also integrally closed.

Theorem I. Suppose R satisfies N_2. Then the polynomial ring $R[x]$ also satisfies N_2.

Proof. Let K denote the quotient field of R and suppose L is a finite extension of $K(x)$, the quotient field of $R[x]$. Let S denote the integral closure of $R[x]$ in L.

Clearly, $R'[x] \subseteq S$. If S is a finite $R'[x]$-module, then since $R'[x]$ is a finite $R[x]$-module (R' is finite over R), S will be a finite $R[x]$-module.

Thus, we may replace R by R' and assume that R is integrally closed. Then $R[x]$ is also integrally closed.

R satisfies N_2 implies R[x] satisfies N_2, continued

If R has characteristic zero, the proof is complete, by Theorem F.

Suppose R has characteristic $p > 0$, i.e., $\mathbb{Z}_p \subseteq R$.

We claim there exists a finite extension K' of K, an exponent $q = p^e$, for some e, and γ in the algebraic closure of L such that $L \subseteq K'(x^{\frac{1}{q}}, \gamma)$ and γ is separable over $K'(x^{\frac{1}{q}})$.

Suppose the claim holds. Let R_0 be the integral closure of R in K'. Then $R_0[x]$ is the integral closure of $R[x]$ in $K'(x)$. Since R_0 is finite over R, $R_0[x]$ is finite over $R[x]$.

If the integral closure of $R_0[x]$ in $K'(x^{\frac{1}{q}}, \gamma)$ is finite over $R_0[x]$, it is finite over $R[x]$. Thus, we may replace K' by K and R_0 by R and assume $K = K'$.

R satisfies N_2 implies R[x] satisfies N_2, continued

So, let S denote the integral closure of $R[x]$ in $K(x^{\frac{1}{q}}, \gamma)$. Let T denote the integral closure of $R[x]$ in $K(x^{\frac{1}{q}})$.

Since $R[x^{\frac{1}{q}}]$ is contained in $K(x^{\frac{1}{q}})$, is integral over $R[x]$, and is integrally closed (it's a polynomial ring over R), we have $T = R[x^{\frac{1}{q}}]$, which is a finite $R[x]$-module.

On the other hand, by the Remark following Theorem F, the integral closure of T in $K(x^{\frac{1}{q}}, \gamma)$, which is S, is a finite T-module. Thus, S is a finite $R[x]$-module, as required.

R satisfies N_2 implies R[x] satisfies N_2, continued

It remains to prove the claim.

For this, we first make an observation.

Let E be a field of characteristic $p > 0$ and suppose β is separable over E, with minimal polynomial $f(y)$. If $E_0 \supseteq E$ is a field containing the qth roots of the coefficients of $f(y)$, then $\beta^{\frac{1}{q}}$ is separable over E_0.

To see this, suppose $f(y) = y^n + e_1 y^{n-1} + \cdots + e_n$, with each $e_j \in E$. If δ is a root of $f(y)$, then $\delta^{\frac{1}{q}}$ is a root of $f_q(y) = y^n + e_1^{\frac{1}{q}} y^{n-1} + \cdots + e_n^{\frac{1}{q}}$.

Since $f(y)$ has distinct roots, $f_q(y)$ has distinct roots, and hence $\beta^{\frac{1}{q}}$ is separable over E_0.

R satisfies N_2 implies R[x] satisfies N_2, continued

To prove the claim, we can write $K(x) \subseteq F \subseteq L$, where F is separable over $K(x)$ and L is purely inseparable over F. There exists $\beta \in F$ such that $F = K(x, \beta)$. Moreover, there exist $\alpha_1, \dots, \alpha_s \in L$ and $q = p^e$ such that $\alpha_i^q \in K(x, \beta)$, for all i, and $L = K(x, \beta, \alpha_1, \dots, \alpha_s)$. Suppose

$$f(y) = y^n + \frac{c_n(x)}{d_n(x)} y^{n-1} + \cdots + \frac{c_n(x)}{d_n(x)}$$

is the minimal polynomial of β over $K(x)$. For each $1 \le i \le s$, we have an equation

$$\alpha_i^q = \frac{a_{0,i}(x)}{b_{0,i}(x)} \cdot 1 + \cdots + \frac{a_{n-1,i}(x)}{b_{n-1,i}(x)} \cdot \beta^{n-1},$$

where the fractions in this equation belong to $K(x)$. Let K' be the field obtained by adjoining the qth roots of the coefficients of all $a_{i,j}(x), b_{i,j}(x), c_i(x), d_i(x)$ to K. Set $\gamma = \beta^{\frac{1}{q}}$.

Then $L \subseteq K'(x^{\frac{1}{q}}, \gamma)$ and K' is a finite extension of K.

By the observation above, γ is separable over $K'(x^{\frac{1}{q}})$. □

The main theorem

We are now ready to prove the main result of this section.

Theorem J. Suppose R is a Nagata ring and T is a finitely generated R-algebra. Then T is a Nagata ring.

Proof. By induction on the number of ring generators of T over R, we may assume that $T = R[x]$, for some $x \in T$. Let $Q \subseteq T$ be a prime ideal. We must show that T/Q satisfies N_2.

Set $q := Q \cap R$. Then $T/Q = R/q[\overline{x}]$, where $\overline{x}$ denotes the image of x in T/Q. Since R/q is a Nagata ring, upon changing notation we are reduced to proving the following statement.

If R is a Nagata ring and T is an integral domain generated as a ring over R by a single element x, then T satisfies N_2.

The main theorem

Suppose x is algebraic over R. If x is integral over R, then T is a Nagata ring, by Comment 5 above. Otherwise, there exists $a \in R$ such that ax is integral over R.

Thus the ring $A := R[ax]$ is a Nagata ring, by Comment 5 above. Moreover, A and T have the same quotient field and $T = A[x]$.

Let E be a finite extension of the quotient field of T (and hence A). Temporarily suspending our $'$ convention, let A' denote the integral closure of A in E and T' denote the integral closure of T in E.

Note that A' is a finite A-module. Set $T_0 := A'[x]$. Then $T' = T_0'$. We will show that T_0' is a finite T_0-module.

If so, since A' is a finite A-module, $A'[x] = T_0'$ is a finite module over $A[x] = T$.

Thus, $T_0' = T'$ is a finite T-module, which is what we want.

The main theorem

To see that T_0' is a finite T_0-module, we use Theorem D. Since $A_a = T_a$, $A'_a = T'_a$. But $A' \subseteq T_0 \subseteq T'$, so $(T_0)_a = T'_a = (T'_0)_a$, and therefore, the first condition in Theorem D holds for T_0.

For the second condition, let $Q \subseteq T_0$ be a maximal ideal (containing a or not). Set $P := Q \cap A'$. Then A'_P is analytically unramified, by Theorem H.

Thus the integral closure of $A'_P[x] = (T_0)_P$ is a finite $(T_0)_P$-module. Since $(T_0)_Q$ is a further localization, $(T'_0)_Q$ is a finite $(T_0)_Q$-module, which is what we want.

Thus, T_0' is finite over T_0, which shows that T satisfies N_2.

If x is algebraically independent over R, then $T = R[x]$ satisfies N_2 by Theorem I. Thus, the proof of Theorem J is complete. □

The main theorem

We now easily recover the geometric case.

Corollary K. Let k be a field and R a finitely generated k algebra. Then R is a Nagata ring. In particular, R satisfies N_2.

Proof. The field k is a Nagata ring, so apply our main theorem.